Lattices with and lattices without spectral gap

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For Fritz Grunewald on his 60th birthday

Abstract

Let $G = \mathbf{G}(\mathbf{k})$ be the $\mathbf{k}$-rational points of a simple algebraic group $\mathbf{G}$ over a local field $\mathbf{k}$ and let Γ be a lattice in G. We show that the regular representation $\rho_{\Gamma\backslash G}$ of G on $L^2(\Gamma\backslash G)$ has a spectral gap, that is, the restriction of $\rho_{\Gamma\backslash G}$ to the orthogonal of the constants in $L^2(\Gamma\backslash G)$ has no almost invariant vectors. On the other hand, we give examples of locally compact simple groups G and lattices Γ for which $L^2(\Gamma\backslash G)$ has no spectral gap. This answers in the negative a question asked by Margulis [Marg91, Chapter III, 1.12]. In fact, G can be taken to be the group of orientation preserving automorphisms of a k-regular tree for $k > 2$.

1 Introduction

Let G be a locally compact group. Recall that a unitary representation π of G on a Hilbert space $\mathcal{H}$ has almost invariant vectors if, for every compact subset Q of G and every $\varepsilon > 0$, there exists a unit vector $\xi \in \mathcal{H}$ such that $\sup_{x\in Q} \|\pi(x)\xi - \xi\| < \varepsilon$. If this holds, we also say that the trivial representation 1_G is weakly contained in π.

Recall that a lattice Γ in G is a discrete subgroup such that there exists a finite G-invariant regular Borel measure μ on $\Gamma\backslash G$. Denote by $\rho_{\Gamma\backslash G}$ the unitary representation of G given by right translation on the Hilbert space $L^2(\Gamma\backslash G, \mu)$ of the square integrable measurable functions on $\Gamma\backslash G$. The subspace $\mathbb{C}1_{\Gamma\backslash G}$ of the constant functions on $\Gamma\backslash G$ is G-invariant as well as its

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orthogonal complement

$$L_0^2(\Gamma\backslash G) = \left\{ \xi \in L^2(\Gamma\backslash G) \,:\, \int_{\Gamma\backslash G} \xi(x) d\mu(x) = 0 \right\}.$$

Denote by $\rho^0_{\Gamma\backslash G}$ the restriction of $\rho_{\Gamma\backslash G}$ to $L_0^2(\Gamma\backslash G, \mu)$. We say that $\rho_{\Gamma\backslash G}$ (or $L^2(\Gamma\backslash G, \mu)$) has a *spectral gap* if $\rho^0_{\Gamma\backslash G}$ has no almost invariant vectors. (In [Marg91, Chapter III., 1.8], Γ is then called weakly cocompact.) It is well-known that $L^2(\Gamma\backslash G)$ has a spectral gap when Γ is cocompact in G (see [Marg91, Chapter III, 1.10]). Margulis (*op.cit*, 1.12) asks whether this result holds more generally when Γ is a subgroup of finite covolume.

The goal of this note is to prove the following results:

Theorem 1 *Let* $\mathbf{G}$ *be a simple algebraic group over a local field* $\mathbf{k}$ *and* $G = \mathbf{G}(\mathbf{k})$, *the group of* $\mathbf{k}$*-rational points in* $\mathbf{G}$. *Let* Γ *be a lattice in* G. *Then the unitary representation* $\rho_{\Gamma\backslash G}$ *on* $L^2(\Gamma\backslash G)$ *has a spectral gap.*

Theorem 2 *For an integer* $k > 2$, *let* X *be the* k*–regular tree and* $G = \mathrm{Aut}(X)$. *Then* G *contains a lattice* Γ *for which the unitary representation* $\rho_{\Gamma\backslash G}$ *on* $L^2(\Gamma\backslash G)$ *has no spectral gap.*

So, Theorem 2 answers in the negative Margulis' question mentioned above.

Theorem 1 is known in case $\mathbf{k} = \mathbf{R}$ ([Bekk98]). It holds, more generally, when G is a real Lie group ([BeCo08]). Observe also that when $\mathbf{k}-\mathrm{rank}(\mathbf{G}) \geq 2$, the group G has Kazhdan's Property (T) (see [BHV]) and Theorem 1 is clear in this case. When $\mathbf{k}$ is non-archimedean with characteristic 0, every lattice Γ in $\mathbf{G}(\mathbf{k})$ is uniform (see [Serr, p.84]) and hence the result holds as mentioned above. By way of contrast, G has many non uniform lattices when the characteristic of $\mathbf{k}$ is non zero (see [Serr] and [Lubo91]). So, in order to prove Theorem 1, it suffices to consider the case where the characteristic of $\mathbf{k}$ is non-zero and where $\mathbf{k} - \mathrm{rank}(\mathbf{G}) = 1$.

Recall that when $\mathbf{k}$ is non-archimedean and $\mathbf{k} - \mathrm{rank}(\mathbf{G}) = 1$, the group $\mathbf{G}(\mathbf{k})$ acts by automorphisms on the associated Bruhat-Tits tree X (see [Serr]). This tree is either the k-regular tree X_k (in which every vertex has constant degree k) or is the bi-partite bi-regular tree X_{k_0,k_1} (where every vertex has either degree k_0 or degree k_1 and where all neighbours of a vertex of degree k_i have degree k_{1-i}). The proof of Theorem 1 will use the special structure of a fundamental domain for the action of Γ on X as described in [Lubo91] (see also [Ragh89] and [Baum03]).

Theorems 1 and 2 provide a further illustration of the different behaviour of general tree lattices as compared to lattices in rank one simple Lie groups over local fields; for more on this topic, see [Lubo95].

The proofs of Theorems 1 and 2 will be given in Sections 3 and 4; they rely in a crucial way on Proposition 6 from Section 2, which relates the existence of a spectral gap with expander diagrams. In turn, Proposition 6 is based, much in the spirit of [Broo81], on analogues for diagrams proved in [Mokh03] and [Morg94] of the inequalities of Cheeger and Buser between the isoperimeric constant and the bottom of the spectrum of the Laplace operator on a Riemannian manifold (see Proposition 5). This connection between the combinatorial expanding property and representation theory is by now a very popular theme; see [Lubo94] and the references therein. While most applications in this monograph are from representation theory to combinatorics, we use in the current paper this connection in the opposite direction: the existence or absence of a spectral gap is deduced from the existence of an expanding diagram or of a non-expanding diagram, respectively.

2 Spectral gap and expander diagrams

We first show how the existence of a spectral gap for groups acting on trees is related with the bottom of the spectrum of the Laplacian for an associated diagram.

A graph X consists of a set of vertices VX, a set of oriented edges EX, a fix-point free involution $^{-} : EX \to EX$, and end point mappings $\partial_i : EX \to VX$ for $i = 0, 1$ such that $\partial_i(\overline{e}) = \partial_{1-i}(e)$ for all $e \in EX$. Assume that X is locally finite, that is, for every $x \in VX$, the degree $\deg(x)$ of x is finite, where $\deg(x)$ is the cardinality of the set

$$\partial_0^{-1}(x) = \{e \in EX \ : \ \partial_0(e) = x\}.$$

The group $\mathrm{Aut}(X)$ of automorphisms of the graph X is a locally compact group in the topology of pointwise convergence on X, for which the stabilizers of vertices are compact open subgroups.

We will consider infinite graphs called diagrams of finite volume. An *edge-indexed graph* (D, i) is a graph D equipped with a function $i : ED \to \mathbf{R}^+$ (see [BaLu01, Chapter 2]). A measure μ for an edge-indexed graph (D, i) is a function $\mu : VD \cup ED \to \mathbf{R}^+$ with the following properties (see [Mokh03] and [BaLu01, 2.6]):

- $i(e)\mu(\partial_0 e) = \mu(e)$
- $\mu(e) = \mu(\overline{e})$ for all $e \in VD$, and
- $\sum_{x \in VD} \mu(x) < \infty$.

Following [Morg94], we will say that $D = (D, i, \mu)$ is a *diagram of finite volume.* The in-degree $\text{indeg}(x)$ of a vertex $x \in VD$ is defined by

$$\text{indeg}(x) = \sum_{e \in \partial_0^{-1}(x)} i(e) = \sum_{e \in \partial_0^{-1}(x)} \frac{\mu(e)}{\mu(x)}.$$

The diagram D is k-regular if $\text{indeg}(x) = k$ for all $x \in VD$.

Let $D = (D, i, \mu)$ be a connected diagram of finite volume. Observe that μ is determined, up to a multiplicative constant, by the weight function i. Indeed, fix $x_0 \in VD$ and set $\Delta(e) = i(e)/i(\overline{e})$ for $e \in ED$. Then

$$\mu(\partial_1 e) = \frac{\mu(\overline{e})}{i(\overline{e})} = \frac{\mu(e)}{i(\overline{e})} = \mu(\partial_0 e)\Delta(e)$$

for every $e \in ED$. Hence $\mu(x) = \Delta(e_1)\Delta(e_2)\ldots\Delta(e_n)\mu(x_0)$ for every path $(e_1, e_2, \ldots, e_n)$ from x_0 to $x \in VD$.

Let $D = (D, i, \mu)$ be a diagram of finite volume. An inner product is defined for functions on VD by

$$\langle f, g \rangle = \sum_{x \in VD} f(x)\overline{g(x)}\mu(x).$$

The Laplace operator Δ on functions f on VD is defined by

$$\Delta f(x) = f(x) - \frac{1}{\text{indeg}(x)} \sum_{e \in \partial_0^{-1}(x)} \frac{\mu(e)}{\mu(x)} f(\partial_1(e)).$$

The operator Δ is a self-adjoint positive operator on $L^2(VD)$. Let

$$L_0^2(VD) = \{f \in L^2(VD) \ : \ \langle f, 1_{VD} \rangle = 0\}$$

and set

$$\lambda(D) = \inf_f \langle \Delta f, f \rangle,$$

where f runs over the unit sphere in $L_0^2(VD)$. Observe that

$$\lambda(D) = \inf\{\lambda \ : \ \lambda \in \sigma(\Delta) \setminus \{0\}\},$$

where $\sigma(\Delta)$ is the spectrum of Δ.

Let now X be a locally finite tree, and let G be a closed subgroup of $\mathrm{Aut}(X)$. Assume that G acts with finitely many orbits on X. Let Γ be a discrete subgroup of G acting without inversion on X. Then the quotient graph $\Gamma\backslash X$ is well-defined. Since Γ is discrete, for every vertex x and every edge e, the stabilizers Γ_x and Γ_e are finite. Moreover, Γ is a lattice in G if and only if Γ is a lattice in $\mathrm{Aut}(X)$ and this happens if and only if

$$\sum_{x \in D} \frac{1}{|\Gamma_x|} < \infty,$$

where D is a fundamental domain of Γ in X (see [Serr]). The quotient graph $\Gamma\backslash X \cong D$ is endowed with the structure of an edge-indexed graph given by the weight function $i : ED \to \mathbf{R}^+$ where $i(e)$ is the index of Γ_e in Γ_x for $x = \partial_0(e)$. A measure $\mu : VD \cup ED \to \mathbf{R}^+$ is defined by

$$\mu(x) = \frac{1}{|\Gamma_x|} \qquad \text{and} \qquad \mu(e) = \frac{1}{|\Gamma_e|}$$

for $x \in VD$ and $e \in ED$. Observe that $\mu(VD) = \sum_{x \in D} 1/|\Gamma_x| < \infty$. So, $D = (D, i, \mu)$ is a diagram of finite volume.

Let G be a group acting on a tree X. As in [BuMo00, 0.2], we say that the action of G on X is *locally ∞-transitive* if, for every $x \in VX$ and every $n \geq 1$, the stabilizer G_x of x acts transitively on the sphere $\{y \in X \ : d(x, y) = n\}$.

Proposition 3 *Let X be either the k-regular tree X_k or the bi-partite bi-regular tree X_{k_0,k_1} for $k \geq 3$ or $k_0 \geq 3$ and $k_1 \geq 3$. Let G be a closed subgroup of* $\mathrm{Aut}(X)$*. Assume that the following conditions are both satisfied:*

- *G acts transitively on VX in the case $X = X_k$ and G acts transitively on the set of vertices of degree k_0 as well as on the set of vertices of degree k_1 in the case $X = X_{k_0,k_1}$;*

- *the action of G on X is locally ∞-transitive.*

Let Γ be a lattice in G and let $D = \Gamma\backslash X$ be the corresponding diagram of finite volume. The following properties are equivalent:

(i) the unitary representation $\rho_{\Gamma\backslash G}$ on $L^2(\Gamma\backslash G)$ has a spectral gap;

(ii) $\lambda(D) > 0$.

For the proof of this proposition, we will need a few general facts. Let G be a second countable locally compact group and U a compact subgroup of G. Let $C_c(U\backslash G/U)$ be the space of continuous functions $f : G \to \mathbf{C}$ which have compact support and which are constant on the double cosets UgU for $g \in G$.

Fix a left Haar measure μ on G. Recall that $L^1(G, \mu)$ is a Banach algebra under the convolution product, the L^1-norm and the involution $f^*(g) = \overline{f(g^{-1})}$; observe that $C_c(U\backslash G/U)$ is a $*$-subalgebra of $L^1(G, \mu)$. Let π be a (strongly continuous) unitary representation of G on a Hilbert space $\mathcal{H}$. A continuous $*$-representation of $L^1(G)$, still denoted by π, is defined on $\mathcal{H}$ by

$$\pi(f)\xi = \int_G f(x)\pi(x)\xi d\mu(x), \qquad f \in L^1(G), \quad \xi \in \mathcal{H}.$$

Assume that the closed subspace $\mathcal{H}^U$ of U-invariant vectors in $\mathcal{H}$ is non-zero. Then $\pi(f)\mathcal{H}^U \subset \mathcal{H}^U$ for all $f \in C_c(U\backslash G/U)$. In this way, a continuous $*$-representation π_U of $C_c(U\backslash G/U)$ is defined on $\mathcal{H}^U$.

Proposition 4 *With the previous notation, let $f \in C_c(U\backslash G/U)$ be a function with the following properties: $f(x) \geq 0$ for all $x \in G$, $\int_G f d\mu = 1$, and the subgroup generated by the support of f is dense in G. The following conditions are equivalent:*

(i) the trivial representation 1_G is weakly contained in π;

(ii) 1 belongs to the spectrum of the operator $\pi_U(f)$.

Proof Assume that 1_G is weakly contained in π. There exists a sequence of unit vectors $\xi_n \in \mathcal{H}$ such that

$$\lim_n \|\pi(x)\xi_n - \xi_n\| = 0,$$

uniformly over compact subsets of G. Let

$$\eta_n = \int_U \pi(u)\xi_n du,$$

where du denotes the normalized Haar measure on U. It is easily checked that $\eta_n \in \mathcal{H}^U$ and that

$$\lim_n \|\pi(f)\eta_n - \eta_n\| = 0.$$

Since

$$\|\eta_n - \xi_n\| \le \int_U \|\pi(u)\xi_n - \xi_n\| du,$$

we have $\|\eta_n\| \ge 1/2$ for sufficiently large n. This shows that 1 belongs to the spectrum of the operator $\pi_U(f)$.

For the converse, assume that 1 belongs to the spectrum of $\pi_U(f)$. Hence, 1 belongs to the spectrum of $\pi(f)$, since $\pi_U(f)$ is the restriction of $\pi(f)$ to the invariant subspace $\mathcal{H}^U$. As the subgroup generated by the support of f is dense in G, this implies that 1_G is weakly contained in π (see [BHV, Proposition G.4.2]).

Proof of Proposition 3 We give the proof only in the case where X is the bi-regular tree X_{k_0,k_1}. The case where X is the regular tree X_k is similar and even simpler.

Let X_0 and X_1 be the subsets of X consisting of the vertices of degree k_0 and k_1, respectively. Fix two points $x_0 \in X_0$ and $x_1 \in X_1$ with $d(x_0, x_1) = 1$. So, X_0 is the set of vertices x for which $d(x_0, x)$ is even and X_1 is the set of vertices x for which $d(x_0, x)$ is odd. Let U_0 and U_1 be the stabilizers of x_0 and x_1 in G. Since G acts transitively on X_0 and on X_1, we have $G/U_0 \cong X_0$ and $G/U_1 \cong X_1$.

We can view the normed $*$-algebra $C_c(U_0\backslash G/U_0)$ as a space of finitely supported functions on X_0. Since U_0 acts transitively on every sphere around x_0, it is well-known that the pair (G, U_0) is a Gelfand pair, that is, the algebra $C_c(U_0\backslash G/U_0)$ is commutative (see for instance [BLRW09, Lemma 2.1]). Observe that $C_c(U_0\backslash G/U_0)$ is the linear span of the characteristic functions $\delta_n^{(0)}$ (lifted to G) of spheres of even radius n around x_0. Moreover, $C_c(U_0\backslash G/U_0)$ is generated by $\delta_2^{(0)}$; indeed, this follows from the formulas (see

[BLRW09, Theorem 3.3])

$$\begin{aligned} \delta_4^{(0)} &= \delta_2^{(0)} * \delta_2^{(0)} - k_0(k_1-1)\delta_0^{(0)} - (k_1-2)\delta_2^{(0)} \\ \delta_{2n+2}^{(0)} &= \delta_2^{(0)} * \delta_{2n}^{(0)} - (k_0-1)(k_1-1)\delta_{2n-2}^{(0)} - (k_1-2)\delta_{2n}^{(0)} \qquad \text{for} \quad n \geq 2. \end{aligned}$$

Let $f_0 = \frac{1}{\|\delta_2^{(0)}\|_1}\delta_2^{(0)}$. We claim that f_0 has all the properties listed in Proposition 4.

Indeed, f_0 is a non-negative and U_0-bi-invariant function on G with $\int_G f_0(x)dx = 1$. Moreover, let H be the closure of the subgroup generated by the support of f_0. Assume, by contradiction, that $H \neq G$. Then there exists a function in $C_c(U_0\backslash G/U_0)$ whose support is disjoint from H. This is a contradiction, as the algebra $C_c(U_0\backslash G/U_0)$ is generated by f_0. This shows that $H = G$.

Let π be the unitary representation of G on $L_0^2(\Gamma\backslash G)$ defined by right translations. Observe that the space of $\pi(U_0)$-invariant vectors is $L_0^2(\Gamma\backslash X_0)$. So, we have a $*$-representation π_{U_0} of $C_c(U_0\backslash G/U_0)$ on $L^2(\Gamma\backslash X_0, \mu)$, where μ is the measure on the diagram $D = \Gamma\backslash X$, as defined above.

Similar facts are also true for the algebra $C_c(U_1\backslash G/U_1)$: this is a commutative normed $*$-algebra, it is generated by the characteristic function $\delta_2^{(1)}$ of the sphere of radius 2 around x_1, and the representation π of G on $L_0^2(\Gamma\backslash G)$ induces a $*$-representation π_{U_1} of $C_c(U_1\backslash G/U_1)$ on $L_0^2(\Gamma\backslash X_1, \mu)$. Likewise, the function $f_1 = \frac{1}{\|\delta_2^{(1)}\|_1}\delta_2^{(1)}$ has all the properties listed in Proposition 4.

Let A_X be the adjacency operator defined on $\ell^2(X)$ by

$$A_X f(x) = \frac{1}{\deg(x)} \sum_{e\in\partial_0^{-1}(x)} f(\partial_1(e)), \qquad f \in \ell^2(X).$$

Since A_X commutes with automorphisms of X, it induces an operator A_D on $L^2(VD, \mu)$ given by

$$A_D f(x) = \frac{1}{\text{indeg}(x)} \sum_{e\in\partial_0^{-1}(x)} \frac{\mu(e)}{\mu(x)} f(\partial_1(e)), \qquad f \in L^2(VD, \mu),$$

where D is the diagram obtained from the quotient graph $\Gamma\backslash X$. So, $\Delta = I - A_D$, where Δ is the Laplace operator on D.

Let B_D denote the restriction of A_D to the space $L^2_0(VD,\mu)$. It follows that $\lambda(\Delta) > 0$ if and only if 1 does not belong to the spectrum of B_D.

Proposition 3 will be proved, once we have shown the following

Claim: 1 belongs to the spectrum of B_D if and only if 1_G is weakly contained in π.

For this, we consider the squares of the operators A_X and A_D and compute

$$A_X^2 f(x) = \frac{1}{k_0k_1}\deg(x)f(x) + \frac{1}{k_0k_1}\sum_{d(x,y)=2} f(y), \qquad f \in \ell^2(X).$$

The subspaces $\ell^2(X_0)$ and $\ell^2(X_1)$ of $\ell^2(X)$ are invariant under A_X^2 and the restrictions of A_X^2 to $\ell^2(X_0)$ and $\ell^2(X_1)$ are given by right convolution with the functions

$$g_0 = \frac{1}{k_0k_1}\delta_e + (1 - \frac{1}{k_0k_1})f_0$$
$$g_1 = \frac{1}{k_0k_1}\delta_e + (1 - \frac{1}{k_0k_1})f_1,$$

where δ_e is the Dirac function at the group unit e of G.

It follows that the restrictions of B_D^2 to the subspaces $L^2_0(\Gamma\backslash X_0,\mu)$ and $L^2_0(\Gamma\backslash X_1,\mu)$ coincide with the operators $\pi_{U_0}(g_0)$ and $\pi_{U_1}(g_1)$, respectively.

For $i = 0, 1$, the spectrum $\sigma(\pi_{U_i}(g_i))$ of $\pi_{U_i}(g_i)$ is the set

$$\sigma(\pi_{U_i}(g_i)) = \left\{ \frac{1}{k_0k_1} + (1 - \frac{1}{k_0k_1})\lambda \ : \lambda \in \sigma(\pi_{U_i}(f_i)) \right\}.$$

Thus, 1 belongs to the spectrum of $\pi_{U_0}(f_i)$ if and only if 1 belongs to the spectrum of $\pi_{U_0}(g_i)$.

To prove the claim above, assume that 1 belongs to the spectrum of B_D. Then 1 belongs to the spectrum of B_D^2. Hence 1 belongs to the spectrum of either $\pi_{U_0}(g_0)$ or $\pi_{U_1}(g_1)$ and therefore 1 belongs to the spectrum of either $\pi_{U_0}(f_0)$ or $\pi_{U_1}(f_1)$. It follows from Proposition 4 that 1_G is weakly contained in π.

Conversely, suppose that 1_G is weakly contained in π. Then, again by Proposition 4, 1 belongs to the spectra of $\pi_{U_0}(f_0)$ and $\pi_{U_1}(f_1)$. Hence, 1 belongs to the spectra of $\pi_{U_0}(g_0)$ and $\pi_{U_1}(g_1)$. We claim that 1 belongs to the spectrum of B_D.

Indeed, assume by contradiction that 1 does not belong to the spectrum of B_D, that is, $B_D - I$ has a bounded inverse on $L^2_0(VD, \mu)$. Since 1 belongs to the spectrum of the self-adjoint operator $\pi_{U_0}(g_0)$, there exists a sequence of unit vectors $\xi_n^{(0)}$ in $L^2_0(\Gamma\backslash X_0, \mu)$ with

$$\lim_n \|\pi_{U_0}(g_0)\xi_n^{(0)} - \xi_n^{(0)}\| = 0.$$

As the restriction of B_D^2 to $L^2_0(\Gamma\backslash X_0, \mu)$ coincides with $\pi_{U_0}(g_0)$, we have

$$\begin{aligned}\|\pi_{U_0}(g_0)\xi_n^{(0)} - \xi_n^{(0)}\| &= \|(B_D^2 - I)\xi_n^{(0)}\| \\ &= \|(B_D - I)(B_D + I)\xi_n^{(0)}\| \\ &\geq \frac{1}{\|(B_D - I)^{-1}\|}\|(B_D + I)\xi_n^{(0)}\|\end{aligned}$$

So, $\lim_n \|B_D\xi_n^{(0)} + \xi_n^{(0)}\| = 0$. On the other hand, observe that B_D maps $L^2_0(\Gamma\backslash X_0, \mu)$ to the subspace $L^2(\Gamma\backslash X_1, \mu)$ and that these subspaces are orthogonal to each other. Hence,

$$\|B_D\xi_n^{(0)} + \xi_n^{(0)}\|^2 = \|B_D\xi_n^{(0)}\|^2 + \|\xi_n^{(0)}\|^2$$

This is a contradiction since $\|\xi_n^{(0)}\| = 1$ for all n. The proof of Proposition 3 is now complete.■

Next, we rephrase Proposition 3 in terms of expander diagrams. Let (D, i, w) be a diagram with finite volume. For a subset S of VD, set

$$E(S, S^c) = \{e \in ED \ : \ \partial_0(e) \in S, \ \partial_1(e) \notin S\}.$$

We say that D is an *expander diagram* if there exists $\varepsilon > 0$ such that

$$\frac{\mu(E(S, S^c))}{\mu(S)} \geq \varepsilon$$

for all $S \subset VD$ with $\mu(S) \leq \mu(D)/2$. The motivation for this definition comes from expander graphs (see [Lubo94]).

We quote from [Mokh03] and [Morg94] the following result which is standard in the case of finite graphs.

Proposition 5 ***([Mokh03], [Morg94])*** *Let (D, i, w) be a diagram with finite volume. Assume that* $\sup_{e\in ED} i(\overline{e})/i(e) < \infty$ *and that* $\sup_{x\in VD} \mathrm{indeg}(x) < \infty$ *The following conditions are equivalent:*

(i) D is an expander diagram;

(ii) $\lambda(D) > 0$.

As an immediate consequence of Propositions 3 and 5, we obtain the following result which relates the existence of a spectral gap to an expanding property of the corresponding diagram.

Proposition 6 *Let X be either the k-regular tree X_k or the bi-partite bi-regular tree X_{k_0,k_1} for $k \geq 3$ or $k_0 \geq 3$ and $k_1 \geq 3$. Let G be a closed subgroup of* $\mathrm{Aut}(X)$ *satisfiying both conditions from Proposition 3. Let Γ be a lattice in G and let $D = \Gamma\backslash X$ be the corresponding diagram of finite volume. The following properties are equivalent.*

(i) The unitary representation $\rho_{\Gamma\backslash G}$ on $L^2(\Gamma\backslash G)$ has a spectral gap;

(ii) D is an expander diagram.

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3 Proof of Theorem 1

Let $G = \mathbf{G}(\mathbf{k})$ be the $\mathbf{k}$-rational points of a simple algebraic group $\mathbf{G}$ over a local field $\mathbf{k}$ and let Γ be a lattice in G. As explained in the Introduction, we may assume that $\mathbf{k}$ is non-archimedean and that $\mathbf{k} - \mathrm{rank}(\mathbf{G}) = 1$. By the Bruhat-Tits theory, G acts on a regular or bi-partite bi-regular tree X with one or two orbits. Moreover, the action of G on X is locally ∞-transitive (see [Chou94, p.33]).

Passing to the subgroup G^+ of index at most two consisting of orientation preserving automorphisms, we can assume that G acts without inversion. Indeed, assume that $L^2(\Gamma \cap G^+\backslash G^+)$ has a spectral gap. If Γ is contained in G^+, then $L^2(\Gamma\backslash G)$ has a spectral gap since G^+ has finite index (see [BeCo08, Proposition 6]). If Γ is not contained in G^+, then $\Gamma \cap G^+\backslash G^+$ may be identified as a G^+-space with $\Gamma\backslash\Gamma G^+ = \Gamma\backslash G$. Hence, 1_{G^+} is not weakly contained in the G^+-representation defined on $L^2_0(\Gamma\backslash G)$.

Let X be the Bruhat-Tits tree associated to G. It is shown in [Lubo91, Theorem 6.1] (see also [Baum03]) that Γ has fundamental domain D in X of the following form: there exists a finite set $F \subset D$ such that $D \setminus F$ is a union of finitely many disjoint rays $r_1, \ldots, r_s$. (Recall that a ray in X is an infinite path beginning at some vertex and without backtracking.) Moreover, for every ray $r_j = \{x_0^j, x_1^j, x_2^j, \dots\}$ in $D \setminus F$, the stabilizer $\Gamma_{x_i^j}$ of x_i^j is contained in the stabilizer $\Gamma_{x_{i+1}^j}$ of x_{i+1}^j for all i.

To prove Theorem 1, we apply Proposition 6. So, we have to prove that D is an expander diagram.

Choose $i \in \{0, 1, \dots\}$ such that, with

$$D_1 = F \cup \bigcup_{j=1}^{s} \{x_0^j, \dots, x_i^j\},$$

we have $\mu(D_1) > 1/2$.

Let S be a subset of D with $\mu(S) \leq \mu(D)/2$. Then $D_1 \nsubseteq S$. Two cases can occur.

•*First case:* $S \cap D_1 = \emptyset$. Thus, S is contained in

$$\bigcup_{j=1}^{s} \{x_{i+1}^j, x_{i+2}^j, \dots\}.$$

Fix $j \in \{1, \dots, s\}$. Let $i(j) \in \{0, 1, \dots\}$ be minimal with the property that $x_{i(j)+1}^j \in S$. Then $e_j := (x_{i(j)+1}^j, x_{i(j)}^j) \in E(S, S^c)$. Observe that $|\Gamma_{x_{l+1}^j}| = \deg(x_l^j)|\Gamma_{x_l^j}|$ for all $l \geq 0$. Let k be the minimal degree for vertices in X (so, $k = \min\{k_0, k_1\}$ if $X = X_{k_0,k_1}$).Then $\mu(x_{l+1}^j) \leq \mu(x_l^j)/k$ for all l and

$$\mu(e_j) = \frac{1}{|\Gamma_{e_j}|} \geq \frac{k}{|\Gamma_{x_{i(j)}^j}|} = k\mu(x_{i(j)}^j).$$

Therefore, we have

$$\begin{aligned}
\frac{\mu(E(S,S^c))}{\mu(S)} &\geq \frac{\sum_{j=1}^{s}\mu(e_j)}{\sum_{j=1}^{s}\mu(\{x^j_{i(j)+1}, x^j_{i(j)+1}, \ldots, \})} \\
&\geq k\frac{\sum_{j=1}^{s}\mu(x^j_{i(j)})}{\sum_{j=1}^{s}\sum_{l=0}^{\infty}\mu(x^j_{i(j)+l})} \\
&\geq k\frac{\sum_{j=1}^{s}\mu(x^j_{i(j)})}{\sum_{j=1}^{s}\mu(x^j_{i(j)})\sum_{l=0}^{\infty}k^{-l}} \\
&= k\frac{\sum_{j=1}^{s}\mu(x^j_{i(j)})}{\frac{1}{1-k^{-1}}\sum_{j=1}^{s}\mu(x^j_{i(j)})} \\
&= k\frac{1}{\frac{1}{1-k^{-1}}} = k-1.
\end{aligned}$$

•*Second case:* $S \cap D_1 \neq \emptyset$. Then there exist $x \in S \cap D_1$ and $y \in D_1 \setminus S$. Since D_1 is a connected subgraph, there exists a path $(e_1, e_2, \ldots, e_n)$ in ED_1 from x to y. Let $l \in \{1, \ldots, n\}$ be minimal with the property $\partial_0(e_l) \in S$ and $\partial_1(e_l) \notin S$. Then $e_l \in E(S, S^c)$. Hence, with $C = \min\{\mu(e) \ : \ e \in ED_1\} > 0$, we have

$$\frac{\mu(E(S,S^c))}{\mu(S)} \geq \frac{C}{\mu(D)}.$$

This completes the proof of Theorem 1.■

4 Proof of Theorem 2

Let (D, i, μ) be a k-regular diagram. By the "inverse Bass–Serre theory" of groups acting on trees, there exists a lattice Γ in $G = \mathrm{Aut}(X_k)$ for which $D = \Gamma\backslash X_k$. Indeed, we can find a finite grouping of (D, i), that is, a graph of finite groups $\mathbf{D} = (D, \mathcal{D})$ such that $i(e)$ is the index of $\mathcal{D}_e$ in $\mathcal{D}_{\partial_0 e}$ for all $e \in ED$. Fix an origin x_0. Let $\Gamma = \pi_1(\mathbf{D}, x_0)$ be the fundamental group of $(\mathbf{D}, x_0)$. The universal covering of $(\mathbf{D}, x_0)$ is the k-regular tree X_k and the diagram D can identified with the diagram associated to $\Gamma\backslash X_k$. For all this, see (2.5), (2.6) and (4.13) in [BaLu01].

In view of Proposition 6, Theorem 2 will be proved once we present examples of k-regular diagrams with finite volume which are not expanders.

An example of such a diagram appears in [Mokh03, Example 3.4]. For the convenience of the reader, we review the construction.

Fix $k \geq 3$ and let $q = k - 1$. For every integer $n \geq 1$, let D_n be the finite graph with $2n + 1$ vertices:

$$\underset{x_1^{(n)}}{\circ} - \underset{x_2^{(n)}}{\circ} - \circ - \cdots \circ - \underset{x_{2n}^{(n)}}{\circ} - \underset{x_{2n+1}^{(n)}}{\circ}$$

Let D be the following infinite ray:

$$\underset{x_0}{\circ} - \underset{x_1}{\circ} - D_1 - \underset{x_2}{\circ} - \underset{x_3}{\circ} - D_2 - \circ - \circ - \cdots - - \underset{x_{2n-2}}{\circ} - \underset{x_{2n-1}}{\circ} - D_n - \circ - \circ \cdots$$

We first define a weight function i_n on ED_n as follows:

- $i_n(e) = 1$ if $e = (x_1^{(n)}, x_2^{(n)})$ or $e = (x_2^{(n)}, x_1^{(n)})$
- $i_n(e) = q$ if $e = (x_m^{(n)}, x_{m+1}^{(n)})$ for m even
- $i_n(e) = 1$ if $e = (x_m^{(n)}, x_{m+1}^{(n)})$ for m odd
- $i_n(e) = q$ if $e = (x_{m+1}^{(n)}, x_m^{(n)})$ for m even
- $i_n(e) = 1$ if $e = (x_{m+1}^{(n)}, x_m^{(n)})$ for m odd.

Observe that $i_n(e)/i_n(\overline{e}) = 1$ for all $e \in ED_n$. Define now a weight function i on ED as follows:

- $i(e) = q + 1$ if $e = (x_0, x_1)$
- $i(e) = q$ if $e = (x_1, x_0)$
- $i(e) = 1$ if $e = (x_m, x_{m+1})$ for $m \geq 1$
- $i(e) = q$ if $e = (x_{m+1}, x_m)$ for $m \geq 1$
- $i(e) = i_n(e)$ if $e \in ED_n$.

One readily checks that, for every vertex $x \in D$,

$$\sum_{e \in \partial_0^{-1}(x)} i(e) = q + 1 = k,$$

that is, (D, i) is k-regular. The measure $\mu : VD \to \mathbf{R}^+$ corresponding to i (see the remark at the beginning of Section 2) is given by

- $\mu(x_0) = 1/(q+1)$
- $\mu(x_{2m-2}) = 1/q^{m-1}$ for $m \geq 2$
- $\mu(x_{2m-1}) = 1/q^m$ for $m \geq 1$
- $\mu(x) = 1/q^n$ if $x \in D_n$.

One checks that, if we define $\mu(e) = i(e)\mu(\partial_0 e)$ for all $e \in ED$, we have $\mu(\overline{e}) = \mu(e)$. Moreover,

$$\mu(D_n) = (2n+1)\frac{1}{q^n}$$

and hence

$$\mu(D) \leq \frac{1}{q+1} + 2\sum_{n \geq 0} \frac{1}{q^n} + \sum_{n \geq 1} \mu(D_n) < \infty.$$

We have also

$$E(D_n, D_n^c) = \{(x_{2n-1}, x_{2n-2}), (x_{2n}, x_{2n+1})\},$$

so that

$$\mu\left(E(D_n, D_n^c)\right) = q\frac{1}{q^n} + \frac{1}{q^n} = \frac{q+1}{q^n}.$$

Hence

$$\frac{\mu\left(E(D_n, D_n^c)\right)}{\mu(D_n)} = \frac{\frac{q+1}{q^n}}{(2n+1)\frac{1}{q^n}} = \frac{q+1}{2n+1}$$

and

$$\lim_n \frac{\mu\left(E(D_n, D_n^c)\right)}{\mu(D_n)} = 0.$$

Observe that, since $\lim_n \mu(D_n) = 0$, we have $\mu(D_n) \leq \mu(D)/2$ for sufficiently large n. This completes the proof of Theorem 2.■.

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